# On the perfect matching association scheme

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# Discrete Mathematics Research Group at URegina

# Perfect matching

## Definition

A **matching** in a graph $G$ is a collection of edges of $G$ that do not have a vertex in common.

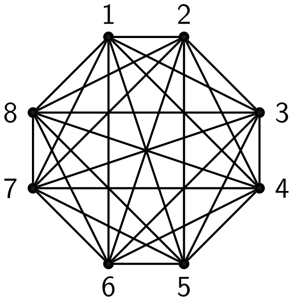


Figure: The complete graph on 8 vertices, $K_8$.

# Perfect matching

**Definition**

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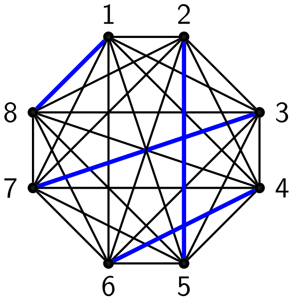



Figure: A matching of $K_8$ (in blue).

# Perfect matching

## Definition

A **perfect matching** in a graph $G$ is a matching that covers every vertex of $G$.

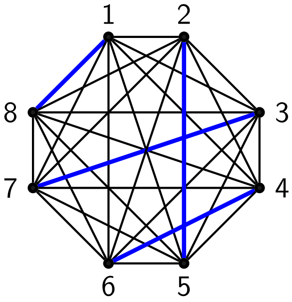



Figure: A perfect matching of $K_8$ (in blue).

# Counting the perfect matchings of $K_{2k}$

**Definition**

Let $M(K_{2k})$ denote the set of all perfect matchings of $K_{2k}$.

**Main goal:** To construct a set of graphs, each with vertex set $M(K_{2k})$, that satisfy a very specific set of constraints.

# Counting the perfect matchings of $K_{2k}$

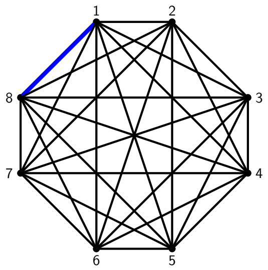



Figure: We select our first edge.

We have:

$$|M(K_{2k})| = \binom{2k}{2} \cdots$$

# Counting the perfect matchings of $K_{2k}$

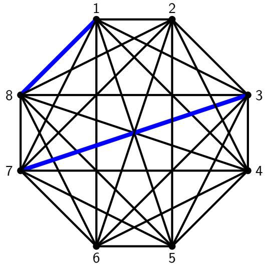



Figure: We select our second edge.

We have:

$$|M(K_{2k})| = \binom{2k}{2}\binom{2k-2}{2}\cdots$$

# Counting the perfect matchings of $K_{2k}$

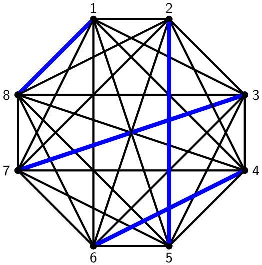



Figure: A perfect matching of $K_8$.

We have:

$$|M(K_{2k})| = \binom{2k}{2}\binom{2k-2}{2}\cdots\binom{2}{2}.$$

# Counting the perfect matchings of $K_{2k}$

We have counted each matching $k!$ times, which means that:

$$|M(2k)| = \frac{1}{k!}\binom{2k}{2}\binom{2k-2}{2}\cdots\binom{2}{2}$$

# Counting the perfect matchings of $K_{2k}$

We have:

$$|M(K_{2k})| = \frac{1}{k!}\binom{2k}{2}\binom{2k-2}{2}\cdots\binom{2}{2}$$
$$= \frac{(2k)(2k-1)(2k-2)(2k-3)\cdots 1}{2^k k!}$$

Recall that $\binom{n}{2} = \frac{n(n-1)}{2}$.

# Counting the perfect matchings of $K_{2k}$

We have:

$$|M(K_{2k})| = \frac{1}{k!}\binom{2k}{2}\binom{2k-2}{2}\cdots\binom{2}{2}$$
$$= \frac{(2k)(2k-1)(2k-2)(2k-3)\cdots 1}{(2k)(2k-2)(2k-4)\cdots 2}$$

Recall tha $\binom{n}{2} = \frac{n(n-1)}{2}$.

# Counting the perfect matchings of $K_{2k}$

We have:

$$\begin{aligned} |M(K_{2k})| &= \frac{1}{k!}\binom{2k}{2}\binom{2k-2}{2}\cdots\binom{2}{2} \\ &= \frac{(2k)(2k-1)(2k-2)(2k-3)\cdots 1}{(2k)(2k-2)(2k-4)\cdots 2} \\ &= (2k-1)(2k-3)(2k-5)\cdots 1 = (2k-1)!! \end{aligned}$$

Recall that $\binom{n}{2} = \frac{n(n-1)}{2}$.

# Adjacencies

Our graphs will each have $(2k-1)!!$ vertices.

**Next step:** to define adjacencies between two vertices.

# Relation between two perfect matchings

We define a relation between two perfect matchings in $M(K_{2k})$.
**Example:** We overlap two perfect matchings of $K_{2k}$.

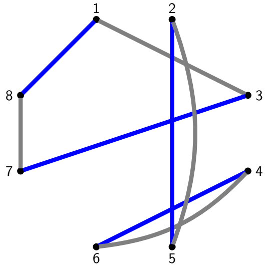



Figure: Two perfect matchings of $M(K_8)$ in grey and blue.

# Relation between two perfect matchings

We define a relation between two perfect matchings in $M(K_{2k})$.
**Example:** This gives rise to a set of cycles of **even** lengths.

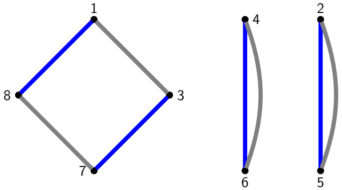



Figure: The union of these two matchings gives us 3 cycles of length 4,2, and 2 respectively.

# Relation between two perfect matchings

## Notation

Let $\lambda \vdash k$ be a partition of $k$ such that $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_t]$. We write $2\lambda = [2\lambda_1, 2\lambda_2, \ldots, 2\lambda_t]$ where $2\lambda \vdash 2k$.

**Example:** If $\lambda \vdash 4$ and $\lambda = [2, 1, 1]$, then $2\lambda = [4, 2, 2]$, where $2\lambda \vdash 8$.

# Building our graphs

**Definition**

Let $P$ and $Q$ be two perfect matchings in $M(K_{2k})$ and $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_t]$ is a partition of $k$. We say that $P$ and $Q$ are $\lambda$-related if $P \cup Q = C_{2\lambda_1} \cup C_{2\lambda_2} \cup \cdots \cup C_{2\lambda_t}$.

**Example:**

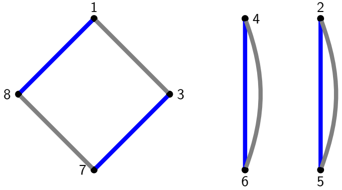



Figure: Our blue and grey perfect matching are [2, 1, 1]-related.

# Constructing a graph

## Definition

The graph $X_\lambda$ is a graph whose vertex set is $M(K_{2k})$. Two vertices, $P$ and $Q$, are adjacent if and only if the corresponding matchings are $\lambda$-related.

Key properties:

- $X_\lambda$ has $(2k-1)!!$ vertices;
- $X_\lambda$ is $d$-regular (each vertex is incident to exactly $d$ edges);
- $X_\lambda$ is vertex transitive with automorphism group $S_{2k}$;
- We have a graph for each partition of $2k$ into even parts.

# Adjacency matrices of a graph

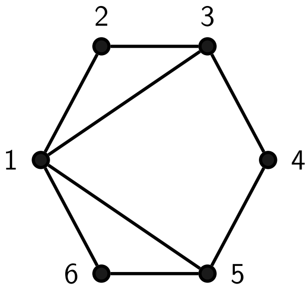
## Definition

Given a graph $X$ with vertex set $V(X)$, the **adjacency matrix of** $X$ is a $V(X) \times V(X)$ matrix with rows and columns indexed by elements of $V(X)$. The coefficients of our matrix are defined as follows:

$$X(u,v) = \begin{cases} 1 & \text{if } u \sim v; \\ 0 & \text{if } u \perp v. \end{cases}$$

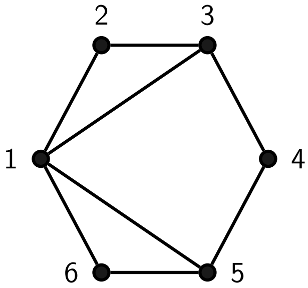
**Example:** Rows and columns of the matrix are indexed by the vertices of our graph.

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | | | | | | |
| 2 | | | | | | |
| 3 | | | | | | |
| 4 | | | | | | |
| 5 | | | | | | |
| 6 | | | | | | |



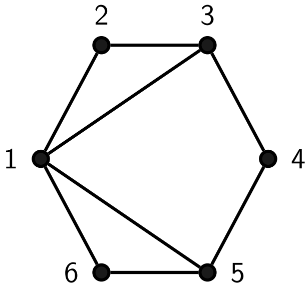
**Example:** $(v_1, v_2) = 1$ if and only if $v_1$ and $v_2$ are adjacent in $X$.

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | | 1 | 1 | | 1 | 1 |
| 2 | 1 | | 1 | | | |
| 3 | 1 | 1 | | 1 | | |
| 4 | | | 1 | | 1 | |
| 5 | 1 | | | 1 | | 1 |
| 6 | 1 | | | | 1 | |



**Example:** $(v_1, v_2) = 0$ otherwise.

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 0 | 1 | 1 |
| 2 | 1 | 0 | 1 | 0 | 0 | 0 |
| 3 | 1 | 1 | 0 | 1 | 0 | 0 |
| 4 | 0 | 0 | 1 | 0 | 1 | 0 |
| 5 | 1 | 0 | 0 | 1 | 0 | 1 |
| 6 | 1 | 0 | 0 | 0 | 1 | 0 |



# Adjacency matrices of a perfect matching graph

## Definition

Let $\lambda \vdash k$. The matrix $A_\lambda$ is a $(2k-1)!! \times (2k-1)!!$ matrix with rows and columns indexed by elements of $M(K_{2k})$. The coefficients of our matrix are defined as follows:

$$X(P,Q) = \begin{cases} 1 & \text{if } P \text{ and } Q \text{ are } \lambda\text{-related} \\ 0 & \text{otherwise} \end{cases}$$

The matrices $A_\lambda$ is a symmetric matrix ($A_\lambda^T = A_\lambda$).

**Example:** We construct the adjacency matrix of $A_\lambda$. Rows and columns are indexed by elements of $M(2k)$.

$$\begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_{(2k-1)!!} \end{array} \begin{array}{c} \begin{array}{ccccc} p_1 & p_2 & p_3 & \cdots & p_{(2k-1)!!} \end{array} \\ \left( \begin{array}{ccccc} & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{array} \right) \end{array}$$

**Example:** Coefficient $(p_i, p_j)$ is 1 if and only if $p_i$ and $p_j$ are $\lambda$-related.

$$
\begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_{(2k-1)!!} \end{array}
\begin{array}{c} \begin{array}{ccccc} p_1 & p_2 & p_3 & \cdots & p_{(2k-1)!!} \end{array} \\
\begin{pmatrix} 0 & 1 & 0 & & 1 \\ 1 & 0 & 1 & & 0 \\ 0 & 1 & 0 & & 0 \\ & & & & \\ 1 & 0 & 0 & & 0 \end{pmatrix} \end{array}
$$

This construction gives rise to $t$ matrices, one for each integer partition of $k$.

# Association schemes

## Definition

A set $\mathcal{A} = \{A_0, A_1, \ldots, A_t\}$ of $v \times v$ binary matrices is an **association scheme** if:

- $A_0 = I_v$ (the identity matrix);
- $\sum_{i=0}^{t} A_i = J$ ($J$ is the all-one matrix);
- $A^T \in \mathcal{A}$; ($A^T$ is the transpose)
- $A_i A_j = c_o A_0 + c_1 A_1 + \ldots + c_t A_t$, where $c_i \in \mathbb{C}$;
- $A_i A_j = A_j A_i$ (matrices commute).

# Perfect matching association schemes

## Definition

The set $\mathcal{A}_{2k} = \{A_{[1,1,\ldots,1]}, A_{[2,1,1,\ldots,1]}, \ldots, A_{[k]}\}$ is known as the perfect matching association scheme.

**Observation:** The set $\mathcal{A}_{2k} = \{A_{[1,1,\ldots,1]}, A_{[2,1,1,\ldots,1]}, \ldots, A_{[k]}\}$ is a set of symmetric matrices that pairwise commute.

**Fact:** A set of symmetric matrices that pairwise commute have the same eigenspaces.

# Eigenspaces

There is an equivalent (and more technical) description of the perfect matching association scheme.

| Eigenspaces | $A_{[1,1,\ldots,1]}$ | $A_{[2,1,1,\ldots,1]}$ | $A_{[3,1,1,\ldots,1]}$ | $\cdots$ | $A_{[k]}$ |
|---|---|---|---|---|---|
| $[2k]$ | | | | | |
| $[2k-2,2]$ | | | | | |
| $[2k-4,4]$ | | | | | |
| $\vdots$ | | | | | |
| $[2,2,2,2,\ldots,2]$ | | | | | |

The eigenspaces of our matrices correspond to irreducible representations of the symmetric group $S_{2k}$ which are $S_{2k}$-modules.

# Eigenvalues

**Question:** Given a $S_{2k}$-module corresponding to $2\mu$, what is the eigenvalue of $A_\lambda$ corresponding to this eigenspace?

| Eigenspaces | $A_{[1,1,\dots,1]}$ | $A_{[2,1,1,\dots,1]}$ | $A_{[3,1,1,\dots,1]}$ | $\cdots$ | $A_{[k]}$ |
|---|---|---|---|---|---|
| $[2k]$ | ? | ? | ? | | ? |
| $[2k-2, 2]$ | ? | ? | ? | | ? |
| $[2k-4, 4]$ | ? | ? | ? | | ? |
| $\vdots$ | ? | ? | ? | | ? |
| $[2, 2, 2, 2 \dots, 2]$ | ? | ? | ? | | ? |

# Eigenvalues

**Question:** Given a $S_{2k}$-module corresponding to $2\mu$, what is the eigenvalue of $A_\lambda$ corresponding to this eigenspace?

| Eigenspaces | $A_{[1,1,\ldots,1]}$ | $A_{[2,1,1,\ldots,1]}$ | $A_{[3,1,1,\ldots,1]}$ | $\cdots$ | $A_{[k]}$ |
|---|---|---|---|---|---|
| $[2k]$ | 1 | ? | ? | | ? |
| $[2k-2,2]$ | 1 | ? | ? | | ? |
| $[2k-4,4]$ | 1 | ? | ? | | ? |
| $\vdots$ | 1 | ? | ? | | ? |
| $[2,2,2,2\ldots,2]$ | 1 | ? | ? | | ? |

# Eigenvalues

**Question:** Given a $S_{2k}$-module corresponding to $2\mu$, what is the eigenvalue of $A_\lambda$ corresponding to this eigenspace?

| Eigenspaces | $A_{[1,1,\ldots,1]}$ | $A_{[2,1,1,\ldots,1]}$ | $A_{[3,1,1,\ldots,1]}$ | $\cdots$ | $A_{[k]}$ |
|---|---|---|---|---|---|
| $[2k]$ | 1 | ✓ | ✓ | ✓ | ✓ |
| $[2k-2,2]$ | 1 | ? | ? | | ? |
| $[2k-4,4]$ | 1 | ? | ? | | ? |
| $\vdots$ | 1 | ? | ? | | ? |
| $[2,2,2,2\ldots,2]$ | 1 | ? | ? | | ? |

The eigenvalues of the $[2k]$-eigenspace corresponds to the degree of each graph (each graph is regular).

# Eigenvalues

**Question:** Given a $S_{2k}$-module corresponding to $2\mu$, what is the eigenvalue of $A_\lambda$ corresponding to this eigenspace?

| Eigenspaces | $A_{[1,1,\ldots,1]}$ | $A_{[2,1,1,\ldots,1]}$ | $A_{[3,1,1,\ldots,1]}$ | $\cdots$ | $A_{[k]}$ |
|---|---|---|---|---|---|
| $[2k]$ | 1 | ✓ | ✓ | ✓ | ✓ |
| $[2k-2,2]$ | 1 | ✓ | ✓ | ✓ | ✓ |
| $[2k-4,4]$ | 1 | ? | ? | | ? |
| $\vdots$ | 1 | ? | ? | | ? |
| $[2,2,2,2\ldots,2]$ | 1 | ? | ? | | ? |

MacDonal (1979) gives formulas for the eigenvalues corresponding to the $[2k-2,2]$-eigenspace.

# Eigenvalues

**Question:** Given a $S_{2k}$-module corresponding to $2\mu$, what is the eigenvalue of $A_\lambda$ corresponding to this eigenspace?

| Eigenspaces | $A_{[1,1,\dots,1]}$ | $A_{[2,1,1,\dots,1]}$ | $A_{[3,1,1,\dots,1]}$ | $\cdots$ | $A_{[k]}$ |
|---|---|---|---|---|---|
| $[2k]$ | 1 | ✓ | ✓ | ✓ | ✓ |
| $[2k-2,2]$ | 1 | ✓ | ✓ | ✓ | ✓ |
| $[2k-4,4]$ | 1 | ✓ | ? | | ? |
| $\vdots$ | 1 | ✓ | ? | | ? |
| $[2,2,2,2\dots,2]$ | 1 | ✓ | ? | | ? |

Diaconis and Holmes (2002) determine all eigenvalues of $A_{[4,2,2,\dots,2]}$.

# Eigenvalues

**Question:** Given a $S_{2k}$-module corresponding to $2\mu$, what is the eigenvalue of $A_\lambda$ corresponding to this eigenspace?

| Eigenspaces | $A_{[1,1,\ldots,1]}$ | $A_{[2,1,1,\ldots,1]}$ | $A_{[3,1,1,\ldots,1]}$ | $\cdots$ | $A_{[k]}$ |
|---|---|---|---|---|---|
| $[2k]$ | 1 | ✓ | ✓ | ✓ | ✓ |
| $[2k-2,2]$ | 1 | ✓ | ✓ | ✓ | ✓ |
| $[2k-4,2,2]$ | 1 | ✓ | ? | | ✓ |
| $\vdots$ | 1 | ✓ | ? | | ✓ |
| $[2,2,2,2\ldots,2]$ | 1 | ✓ | ? | | ✓ |

MacDonald (1979) provides a formula for computing eigenvalues of $A_{[2k]}$.

# Matrix of Eigenvalues

| Eigenspaces | $A_{[1,1,1,]}$ | $A_{[2,1,1]}$ | $A_{[2,2]}$ | $A_{[3,1]}$ | $A_{[4]}$ |
|---|---|---|---|---|---|
| [8] | 1 | 12 | 12 | 32 | 48 |
| [6, 2] | 1 | 5 | -2 | 4 | -8 |
| [4, 4] | 1 | 2 | 7 | -8 | -2 |
| [4, 2, 2] | 1 | -1 | -2 | -2 | 4 |
| [2, 2, 2, 2] | 1 | -6 | 3 | 8 | -6 |

For small $k$, we are able to compute all eigenvalues.

# Further background

- The eigenvalues of the perfect matching association schemes are always integers.
- Godsil and Meagher have derived a formula for computing the eigenvalues using the eigenvectors.
- Srinivasan (2020) developed an inductive algorithm that derives explicit formulas for the eigenvalues of each column.

# Conjecture

It is well-known that the largest eigenvalue occurs on the $[2k]$-eigenspace for each $A_\lambda$ and that this eigenvalue corresponds to the degree of each graph.

## Conjecture (Meagher)

*If $\lambda$ contains at least one part of length 1, then the second highest eigenvalue of $A_\lambda$ occurs on the $[2k-2,2]$-eigenspace.*

# Conjecture

| Eigenspaces | $A_{[1,1,\ldots,1]}$ | $A_{[2,1,1,\ldots,1]}$ | $A_{[3,1,1,\ldots,1]}$ | $\cdots$ | $A_{[k]}$ |
|---|---|---|---|---|---|
| $[2k]$ | 1 | ✓ | ✓ | ✓ | ✓ |
| $[2k-2,2]$ | 1 | | | | ✓ |
| $[2k-4,2,2]$ | 1 | ✓ | ? | | ✓ |
| $\vdots$ | 1 | ✓ | ? | | ✓ |
| $[2,2,2,2\ldots,2]$ | 1 | ✓ | ? | | ✓ |

# The trace trick

**Theorem (GHLMM (2025+))**

*The second largest eigenvalue of $A_{[k-1,1]}$ and $A_{[k-2,1,1]}$ occurs on the occurs on the $[2k-2,2]$-eigenspace.*

**Facts:**

- The trace of a matrix is the sum of its eigenvalues.
- If $A$ is the adjacency matrix of a graph $X$, the trace of $A^2$ is twice the number of edges of $X$.

# The trace trick

**Proof:** The degree of $A_\lambda$ is $\zeta_{[2k]}$, the eigenvalue of the $[2k]$-eigenspace. We see that the trace of $A_\lambda^2$ is:

$$trace(A_\lambda^2) = \sum_{\mu \vdash k} m_{2\mu} \zeta_{2\mu}^2$$

$$trace(A_\lambda^2) = (2k-1)!!\zeta_{[2k]}$$

where the $\zeta_{2\mu}$ are the eigenvalues of $A_\lambda$ occurring with multiplicity $m_{2\mu}$ on the $2\mu$-eigenspace. This means that

$$\textstyle\sum_{\mu \vdash k} m_{2\mu} \zeta_{2\mu}^2 = (2k-1)!!\zeta_{[2k]}.$$

## The trace trick

**Proof:** We know the eigenvalues of the $\mu$-eigenspace for $\mu \in \{[2k], [2k-2, 2]\}$. Thus,

$$\sum_{\mu \notin \{[2k],[2k-2,2]\}} m_{2\mu}\zeta_{2\mu}^2 = (2k-1)!!\zeta_{[2k]} - \zeta_{[2k]}^2 - m_{[2k-2,2]}\zeta_{[2k-2,2]}^2$$

where the $\zeta_{2\mu}$ are the eigenvalues of $A_\lambda$ occurring with multiplicity $m_{2\mu}$.

# The trace trick

**Proof:** Since every element in the sum on left-hand side is a positive integer, we have

$$\zeta_{2\mu}^2 \leqslant (2k-1)!!\zeta_{[2k]} - m_{[2k-2,2]}\zeta_{[2k-2,2]}^2 - \zeta_{[2k]}$$

where the $\zeta_{2\mu}$ correspond to $2\mu$-eigenspaces such that $2\mu \notin \{[2k], [2k-2,2]\}$.

# The trace trick

**Proof:** We then have an upper-bound for

$$\zeta_{\mu}^{2} \leqslant (2k-1)!! - \zeta_{[2k]}^{2} - m_{[2k-2,2]}\zeta_{[2k-2,2]}^{2} - \zeta_{[2k]}$$

where the $\zeta_{2\mu}$ correspond to $2\mu$-eigenspaces such that $2\mu \notin \{[2k],[2k-2,2]\}$.

# The trace trick

**Proof:** We know how to compute the eigenvalues for the $[2k-2,2]$-eigenspace and the $[2k]$-eigenspace. This means that we can bound the $\zeta_\mu^2$ by some function $f$ of $\zeta_{[2k]}$ and $\zeta_{[2k-2,2]}$):

$$\zeta_\mu^2 \leqslant f(\zeta_{[2k]}, \zeta_{[2k-2,2]})$$

The crux is to show that $f(\zeta_{[2k]}, \zeta_{[2k-2,2]}) \leqslant \zeta_{[2k-2,2]}^2$ which then implies that

$$\zeta_\mu^2 \leqslant \zeta_{[2k-2,2]}^2.$$

We show that $f(\zeta_{[2k]}, \zeta_{[2k,2]}) \leqslant \zeta_{[2k-2,2]}^2$ for matrices $A_{[k-1,1]}$ and $A_{[k-2,1,1]}$.

# The trace trick

**Proof:** Because $\zeta_{[2k-2,2]} \geqslant 0$, we then see that

$$\zeta_\mu \leqslant \zeta_{[2k-2,2]}$$

for all $\mu \notin \{[2k], [2k-2, 2]\}$ and the claim follows.

# Results

Using formulas obtained from Srinivasan, we are also able to affirm Meagher's conjecture for three other matrices in the scheme.

**Theorem (GHLMM (2025+))**

*The second largest eigenvalue of* $A_{[4,2,2,\ldots 2]}$, $A_{[4,4,2\ldots,2]}$, $A_{[6,2\ldots,2]}$ *occurs on the* $[2k-2,2]$*-eigenspace.*

# Future work

- What are the diameters of the graphs in $\mathcal{A}(M_{2k})$?
- What is the chromatic number of the graphs in $\mathcal{A}(M_{2k})$?
- Can our methods be further extended to affirm our conjecture on the second highest eigenvalue?